Stability Analysis of Interconnected Deformable Bodies with Closed-Loop Configuration

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The rotational equations of a system of interconnected deformable bodies with closed-loop configuration are derived from the virtual work principle. They yield equilibrium conditions and after linearization necessary and sufficient stability criteria are obtained at least in presence of complete damping.

Introduction

In a previous paper, the dynamics and the stability of a system of interconnected deformable bodies in a topological tree configuration were investigated.[1] The results obtained in that paper will be extended to systems including closed loops of bodies according to ideas developed in a previous communication.[2] At first glance, the general model presented here may appear to be academic but it is of immediate practical importance, and it may be used to investigate the dynamical behavior of mechanisms, manipulators, spacecraft, and all kinds of vehicles.

The problem has not been treated as such in the literature, but one must mention a paper by Ossenberg-Franzes[3] which considers interconnected closed loops of rigid bodies when there exists at least one translational degree of freedom in each loop. In the present paper, no such restriction has been assumed. The system is put into correspondence with a topological graph, the bodies being considered as vertices and the joints as oriented arcs. The now classical tree configuration can be recovered when a number of arcs equal to the cyclomatic number of the graph are cut. The forces and torques in the cut arcs which are necessary to satisfy the closed-loop constraints will be considered as additional unknowns and a complete system of equations (partially differential, partially algebraic) will be obtained by adjoining the kinematical loop equations to the dynamical equations of the associated tree.

These equations can be derived from any standard method, but, as in Ref. 1, we preferred to use the Alembert virtual work principle as it permits us to directly derive the equations in a suitable form for our stability investigation. By an appropriate treatment of the constraint equations, these can also lead to a purely dynamical system of equations.

Complementary System Description

In this paper the graph associated with the system contains closed loops and is not necessarily planar. Even if the incidence matrices S and $\bar{S}$ can still be defined as previously, the matrix T is now indeterminate for elements belonging to a closed loop. If we cut a number of arcs equal to the cyclomatic number of the graph, we can associate a topological tree to the system, and for this tree the matrix T can be defined in terms of the corresponding elements τ^a.

To make a clear difference between the arcs of the associated tree and the arcs that have been cut, the indices a, b, c,... will be used for the arcs of the tree and the indices γ_1, γ_2, γ_3, ... for the arcs that have been cut. Each cut arc γ_i corresponds to a unique loop also denoted γ_i, which consists of this arc γ_i and a path of the tree (Fig. 1). It may be noted that these loops are kinematically independent and that the kinematical relations relative to any other loop of the graph may be expressed as a linear combination of the similar relations pertaining to the previously mentioned set of loops. Denoting by γ^* the body of the loop which, in the associated tree, is on the path between the reference body and all the other bodies of loop γ, useful additional graph arrays can be constructed.

An element $R^{a\gamma}$ of the matrix R is equal to 1 (-1) when arc a belongs to loop γ and has the same (opposite) orientation in loop as arc γ. This element vanishes when a does not belong to loop γ (see Fig. 2).

For arcs a and b of the loop γ, the element $P^{ab\gamma}$ of the tridimensional array P: a) is equal to 1 if b is on the minimum path between a and γ which does not contain γ^*, b) to -1 if b is on the minimum path between a and γ^* which does not contain γ, and c) to zero when a and b coincide and in all other cases. In similar conditions, the element $Q^{a\gamma b}$ of the array Q is equal to 1 when a and b are on two different minimum paths between γ and γ^* and zero otherwise. If a or/and b do (es) not belong to the loop γ the elements $P^{ab\gamma}$ and $Q^{a\gamma b}$ are equal to zero.

From these definitions, it can be easily checked that the following relations hold:

$$R^{a\gamma}\tau^b P^{ab\gamma} = R^{b\gamma}\tau^a P^{ab\gamma} \tag{1a}$$

$$R^{a\gamma}\tau^b Q^{a\gamma b} = -R^{b\gamma}\tau^a Q^{a\gamma b} \tag{1b}$$

$$\sum_i T^{ai} S^{i\gamma} = -R^{a\gamma} \tag{1c}$$

This last relation can be used to check the input graph matrix R and can be related to previous results The matrix T is the left pseudo-inverse of S for the graph; for graphs including closed loops the relation between S and the matrix T of the associated tree becomes $TS = [E, -R]$. A fourth relation involving the modified mass matrices of the tree will also be used. This relation is

$$\sum_i S^{i\gamma} m^{ia} = -m R^{a\gamma}$$

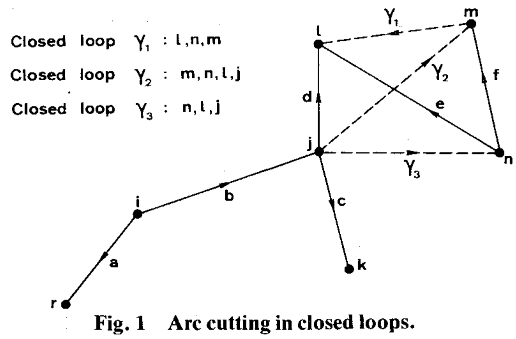

Fig. 1 Arc cutting in closed loops.

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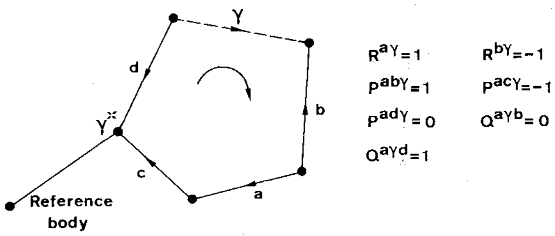

Fig. 2 Example of graph arrays for a closed loop.

Indeed if a belongs to the loop, for every i,

$$S^{i\gamma} m^{ia}_{*} = -m^{ia} R^{a\gamma} \tag{2}$$

and if a does not belong to the loop, the modified mass is the same for the two bodies adjacent to γ, and $R^{a\gamma}$ is equal to zero.

Equations of Motion

Our goal is to obtain the equations of motion in a classical form that is suitable for simulation and stability analysis. The presence of loops implies kinematical constraints, and we will derive a system which is equivalent to the Lagrange equations of the system.

To somewhat simplify the formalism and also to restrict the total number of equations to be considered, we will assume that the cut arcs are rigid, i.e., that the corresponding relative translations and rotations, as well as their virtual changes, are identically equal to zero. These variables will not appear in the equations and further the kinematical constraints of the loops will thus be expressed in terms of the variables defined for the associated tree. It should be noted that we can assume the existence of such a rigid joint without lack of generality. Indeed if the physical system does not include such a joint it can be added together with an additional massless body.

To maintain the equivalence between the original system and the tree configuration, the forces and torques in the cut arcs must be considered as external forces and torques and they will appear in the equations as additional unknowns. These forces and torques act with opposite signs on the bodies adjacent to γ and their values are thus given by $S^{i\gamma}\boldsymbol{F}^{\gamma}$ and $S^{i\gamma}$ $\boldsymbol{L}^{\gamma}$. They do not produce work if the constraints are satisfied, and the complete set of equations will consist of the equations of the tree and the constraint equations of the loops. They form a set of equations with an equal number of unknowns and can then be solved.

The constraints being algebraic, in principle, can be solved and by using their solution in the virtual work principle, the Lagrange equations of the complete system are obtained. Except in particular cases, such as presence of free joints or of hinge connected two-dimensional systems, the solution of the kinematical constraints presents difficulties at the level of computation, and here this last step will be avoided in the derivation of nonlinear equations. In the case of linearized equations the problem will turn out to be quite simple and the Lagrange equations can then be derived for any system.

The virtual work principle can be expressed as

$$\begin{aligned}
&\sum_i \int_i \sigma^{iT}\delta\epsilon^i dv + \sum_a \boldsymbol{F}^a \cdot [\hat{X}^a_\alpha]^T \delta\boldsymbol{z}^a \\
&+ \sum_a \boldsymbol{L}^a \cdot \delta\boldsymbol{\gamma}^a + \sum_i \int_i [-\boldsymbol{f}^i + (\ddot{\boldsymbol{R}} + \ddot{\boldsymbol{\rho}}^i + \ddot{\boldsymbol{p}}^i)] \\
&\cdot \delta(\boldsymbol{R} + \boldsymbol{\rho}^i + \boldsymbol{p}^i)\, dm + \sum_{i,s} \dot{\boldsymbol{h}}^{is} \cdot (\delta\boldsymbol{\psi}^i + \delta\boldsymbol{\gamma}^{is}) \\
&- \sum_\gamma \boldsymbol{F}^\gamma \cdot \sum_i \delta(\boldsymbol{R} + \boldsymbol{\rho}^i + \boldsymbol{p}^{i\gamma} + \bar{S}^{i\gamma}\boldsymbol{z}^\gamma_0) S^{i\gamma} \\
&- \sum_\gamma \boldsymbol{L}^\gamma \cdot \sum_i (\delta\boldsymbol{\psi}^i + \delta\boldsymbol{\gamma}^{i\gamma}) S^{i\gamma} = 0
\end{aligned} \tag{3}$$

In Eq. (3) we have included $\boldsymbol{z}^\gamma_0$ as it can be interesting to consider constant distances between the attachment points of two adjacent bodies even if there is no relative translation in the corresponding joint.

For an equivalent tree configuration, all the virtual displacements can still be expressed in terms of the independent variables $\delta\boldsymbol{R}$, $\delta\boldsymbol{\psi}$, $\delta\boldsymbol{\gamma}^a$, $\delta\boldsymbol{z}^a$, $\delta\beta^i$, the total number of variables being equal to the total number of degrees of freedom of the tree and also of the physical configuration.

The equations of translation and rotation of the complete system are obtained by equating to zero the coefficients of $\delta\boldsymbol{R}$ and $\delta\boldsymbol{\psi}$. As expected these equations do not explicitly contain terms due to presence of closed loops but such terms are present in the other equations. Expressing the various virtual displacement in terms of the virtual changes of the variables, the additional terms due to the closed loops are

$$\begin{aligned}
&- \sum_{i,\gamma} S^{i\gamma} \Big(\sum_j T^{aj}\boldsymbol{b}^{ji} + T^{ai}\boldsymbol{p}^{i\gamma} + T^{ai}\bar{S}^{i\gamma}\boldsymbol{z}^\gamma_0 \Big) \times \boldsymbol{F}^\gamma \\
&\qquad - \sum_{i,\gamma} S^{i\gamma} T^{ai} \boldsymbol{L}^\gamma
\end{aligned}$$

for the rotational equation of joint a.

Using the relations (1), these terms can simply be written as

$$\sum_\gamma R^{a\gamma} (\boldsymbol{\ell}^{a\gamma} \times \boldsymbol{F}^\gamma + \boldsymbol{L}^\gamma)$$

where $\boldsymbol{\ell}^{a\gamma}$ is the vector joining the tip of $\boldsymbol{z}^a$ to the tip of $\boldsymbol{z}^\gamma_0$.

Similarly, the additional terms of the translational equations of the joint a are

$$- \sum_{i,j,\gamma} S^{i\gamma} \bar{S}^{ja} (m^{*}_{ia}/m) \boldsymbol{F}^\gamma$$

or using Eq. (2)

$$\sum_\gamma R^{a\gamma} \boldsymbol{F}^\gamma$$

Finally in the deformation equations these terms are

$$\begin{aligned}
&\sum_{\gamma,a} S^{ka} \left[\frac{\partial \boldsymbol{\alpha}^{ka}}{\partial \beta^k} \right]^T \underline{\Phi}^{ka^T} [\hat{X}^k_\alpha] \cdot \boldsymbol{R}^{a\gamma} (\boldsymbol{\ell}^{a\gamma} \times \boldsymbol{F}^\gamma + \boldsymbol{L}^\gamma) \\
&- \sum_{i,\gamma a} S^{i\gamma} S^{ka} (m^{ia}_{*}/m) \left[\frac{\partial \underline{u}^{ka}}{\partial \beta^k} \right]^T [\hat{X}^k_\alpha] \cdot \boldsymbol{F}^\gamma \\
&+ \sum_{i,\gamma,a} S^{i\gamma} \bar{S}^{ka} (m^{ia}_{*}/m) \left[\frac{\partial \underline{\alpha}^{ka}}{\partial \beta^k} \right]^T \underline{\Phi}^{ka^T} [\hat{X}^k_\alpha] \cdot \boldsymbol{F}^\gamma \times \boldsymbol{z}^a \\
&+ \sum_\gamma \dot{S}^{k\gamma} \left[\frac{\partial \underline{u}^{k\gamma}}{\partial \beta^k} \right]^T [\hat{X}^k_\alpha] \cdot \boldsymbol{F}^\gamma \\
&+ \sum_\gamma S^{k\gamma} \left[\frac{\partial \underline{\alpha}^{k\gamma}}{\partial \beta^k} \right]^T \underline{\Phi}^{k\gamma^T} [\hat{X}^k_\alpha] \cdot (\bar{S}^{k\gamma}\boldsymbol{z}^\gamma \times \boldsymbol{F}^\gamma + \boldsymbol{L}^\gamma)
\end{aligned}$$

Introducing these results, the equation of motion can now be written as

$$\begin{aligned}
&\sum_i \int_i \boldsymbol{p}^i \times \ddot{\boldsymbol{p}}^i dm + \sum_{i,j,k} m^j \boldsymbol{b}^{ij} \times \ddot{\boldsymbol{b}}^{kj} \\
&+ \sum_{i,s} \dot{\boldsymbol{h}}^{is} - \sum_i \int_i \boldsymbol{p}^i \times \boldsymbol{f}^i dm - \sum_{i,j} \boldsymbol{b}^{ij} \times \boldsymbol{F}^j = 0
\end{aligned} \tag{4a}$$

$$\begin{aligned}
&- \sum_i T^{bi} \int_i \boldsymbol{p}^i \times \ddot{\boldsymbol{p}}^i dm - \sum_{i,j,k} T^{bi} m^j \boldsymbol{b}^{ij} \times \ddot{\boldsymbol{b}}^{kj} \\
&\quad - \sum_{i,s} T^{bi} \dot{\boldsymbol{h}}^{is} + \sum_i T^{bi} \int_i \boldsymbol{p}^i \times \boldsymbol{f}^i dm \\
&+ \sum_{i,j} T^{bi} \boldsymbol{b}^{ij} \times \boldsymbol{F}^j - \sum_\gamma R^{b\gamma} (\boldsymbol{L}^\gamma + \boldsymbol{\ell}^{b\gamma} \times \boldsymbol{F}^\gamma) = -\boldsymbol{L}^b
\end{aligned} \tag{4b}$$

$$\sum_i m^{ia}_{\cdot}(\ddot{b}^{ia}+\frac{1}{m}F^i)-\sum_\gamma R^{a\gamma}F^\gamma=-F^a \tag{4c}$$

$$\int_i\left[\frac{\partial\epsilon^i}{\partial\beta^i}\right]^T\sigma^i dv-\int_i\left[\frac{\partial\underline{u}^i}{\partial\beta^i}\right]^T[\hat{X}^i_\alpha]\cdot(f^i-\ddot{p}^i)dm$$
$$+\sum_s\left[\frac{\partial\underline{\alpha}^{is}}{\partial\beta^i}\right]\underline{\Phi}^{is^T}[\hat{X}^i_\alpha]\cdot\dot{h}^{is}$$
$$-\sum_a S^{ia}\left[\frac{\partial\underline{u}^{ia}}{\partial\beta^i}\right]^T[\hat{X}^i_\alpha\cdot F^a$$
$$-\sum_a S^{ia}\left[\frac{\partial\underline{\alpha}^{ia}}{\partial\beta^i}\right]^T\underline{\Phi}^{ia^T}[\hat{X}^i_\alpha]\cdot(L^a+\bar{S}^{ia}z^a\times F^a)$$
$$-\sum_\gamma S^{i\gamma}\left[\frac{\partial\underline{u}^{i\gamma}}{\partial\beta^i}\right]^T[\hat{X}^i_\alpha]\cdot F^\gamma$$
$$-\sum_\gamma S^{i\gamma}\left[\frac{\partial\underline{\alpha}^{i\gamma}}{\partial\beta^i}\right]^T\underline{\Phi}^{i\gamma^T}[\hat{X}^i_\alpha]\cdot(L^\gamma+\bar{S}^{i\gamma}z^\gamma\times F^\gamma)=0 \tag{4d}$$

The equations corresponding to the virtual changes $\delta\underline{\gamma}^a$ and $\delta\underline{z}^a$ have an immediate physical interpretation. They are the rotational and translational equations of the subsystem located "behind" z^a, when joint a (and γ) has been cut. For such a system, F^a and L^a are "external" forces and torques, and these equations thus represent the balance of forces and torques about joint a for such a subsystem. The constraint equations can be simply written as

$$\ell^{\gamma\gamma}=0 \tag{5}$$

$$A^{\gamma\gamma}=E \tag{6}$$

The first constraint implies that the vector joining the two extremities of joint γ (and expressed in terms of the corresponding tree parameters and variables) is equal to zero. The second implies that the product of the various frame transformation matrices encountered in the loop is an identity matrix. The first constraint can easily be solved for some of the components of z^a vectors but the second constraint can only be solved in terms of a particular transformation matrix.

Equilibrium Configuration and Linearized Equations

Here too we will say the system is in equilibrium when its reference frame has the same angular velocity ω_0 as the nominal frame and when all the other variables, including the forces and torques in the cut arcs, are constant. The values of all the vectors and matrices evaluated at this state will be denoted by the subscript o.

In matrix notation and with the previously adopted conventions, the equilibrium conditions can now be written as

$$\tilde{\omega}_0 I_0\omega_0+\tilde{\omega}_0 h_0-\sum_i \tilde{\rho}^i_0 F^i_0-\sum_i\int_i \tilde{x}^i f^i_0\,dm=0 \tag{7}$$

$$-\sum_k T^{ak}\int_k \tilde{x}^k\tilde{\omega}_0\tilde{\omega}_0 x^k dm-\sum_{i,j,k} T^{ai}m^k\tilde{b}^{ik}_0\tilde{\omega}_0\mathrm{auau}\tilde{\omega}\,{}_o b_0{}^{jk}-\sum_{k,r} T^{ak}\tilde{\omega}_0 h^{kr}_0$$
$$+\sum_{i,k} T^{ak}\tilde{b}^{ki}_0 F^i_0+\sum_i T^{ai}\int_i \tilde{x}^i f^i_0 dm$$
$$-\sum_\gamma R^{a\gamma}(L^\gamma_0+\ell^{\tilde{a}}_0 F^\gamma_0)=-L^a_0 \tag{8}$$

$$\sum_i m^{ia}_{\cdot}(\tilde{\omega}_0\tilde{\omega}_0 b^{ia}_0+(1/m)F^i_0)-\sum_\gamma R^{a\gamma}F^\gamma_0=-F^a_0 \tag{9}$$

$$\int_i\left[\frac{\partial\epsilon^i}{\partial\beta^i}\right]^T_0\sigma^i_0 dv-\int_i W^{i^T}_0 f^i_0\,dm+\int_i W^{i^T}_0\tilde{\omega}_0\tilde{\omega}_0 x^i dm+\sum_r V^{ir^T}_0\tilde{\omega}_0 h^{ir}_0$$
$$-\sum_a S^{ia}W^{ia^T}_0 F^a_0-\sum_a S^{ia}V^{ia^T}_0(L^a_0+\bar{S}^{ia}\tilde{z}^a_0 F^a_0)-\sum_\gamma S^{i\gamma}W^{i\gamma^T}_0 F^\gamma_0$$
$$-\sum_\gamma S^{i\gamma}V^{i\gamma^T}_0(L^\gamma_0+\bar{S}^{i\gamma}\tilde{z}^\gamma_0 F^\gamma_0)=0 \tag{10}$$

$$A^{\gamma\gamma}_0=E \tag{11}$$

$$\ell^{\gamma\gamma}_0=0 \tag{12}$$

This set of algebraic equations permits the equilibrium synthesis of the system to perform.

To obtain the equations in a form suitable for stability analysis, the linearized matricial Lagrange equations of the corresponding tree, and the linearized constraint equations are derived. The dynamical equations are obtained from the results of Ref. 1 and the constraints are written under the form

$$-\sum_{a,i} R^{a\gamma}S^{ia}b^{ia}-\sum_i S^{i\gamma}b^{i\gamma}=0 \tag{13}$$

$$-\sum_b R^{b\gamma}\theta^b-\sum_{b,k}R^{b\gamma}S^{kb}V^{kb}_0\beta^k-\sum_k S^{k\gamma}V^{k\gamma}_0\beta^k=0 \tag{14}$$

where b^{ia} and $b^{i\gamma}$ still have to be developed in terms of tree variables.

To prove the symmetry of the results and to obtain simplified expressions, the loop relations (13) and (14) are combined with some of the terms of the equation of the tree.

In particular, we make use of the relations

$$|R^{b\gamma}|\ell^{b\gamma}=|R^{b\gamma}|\{-\tfrac{1}{2}\sum_{a,i}(P^{ba\gamma}+Q^{a\gamma b})\tau^a S^{ia}b^{ia}-\tfrac{1}{2}\sum_i|S^{ib}|b^{ib}+\tfrac{1}{2}\sum_i|S^{i\gamma}|b^{i\gamma}\}$$

[where the expression of $\ell^{b\gamma}$ in the tree has been combined with Eq. (5)] and of the relation

$$|R^{b\gamma}|A^{rb}_0A^{b\gamma}=|R^{b\gamma}|\{E+\tfrac{1}{2}\sum_a(\tau^aP^{ab\gamma}-\tau^aQ^{a\gamma b}+\delta^{ab})\tilde{\theta}^a+\tfrac{1}{2}$$
$$\sum_{a,k}S^{ka}(\tau^aP^{ab\gamma}-\tau^aQ^{a\gamma b}+\delta^{ab})[V^{ka}_0\beta^k]^{\sim}$$
$$+\tfrac{1}{2}\sum_k|S^{k\gamma}|[V^{k\gamma}_0\beta^k]^{\sim}$$
$$-\sum_k\bar{S}^{kb}[V^{kb}_0\beta^k]\tilde{\,}A^{k\gamma}_0$$

where the expression of $\ell^{b\gamma}$ in the tree has been combined with the "tilde" version of Eq. (14).

For the same reason, vanishing terms are added to the left hand side of the deformation equations; these terms are

$$\tfrac{1}{2}\sum_\gamma|S^{i\gamma}|V^{i\gamma^T}_0[(A^{\gamma\gamma}-E)L^\gamma_0+\tilde{\ell}^{\gamma\gamma}F^\gamma_0]+\tfrac{1}{2}\sum_\gamma|S^{i\gamma}|(W^{i\gamma^T}_0+\bar{S}^{i\gamma}V^{i\gamma^T}_0\tilde{z}^\gamma_0)\;(A^{\gamma\gamma}-E)F^\gamma_0\equiv 0$$

This identity being trivial from Eqs. (5) and (6).

If the constraints are written in a differential form, it can be seen from the structure of the equations that F^{γ} and L^{γ} are the Lagrange multipliers corresponding to Eqs. (5) and (6), respectively.

If x_I is the state vector,

$$x_I = [\theta^T \theta^{a^T} z^{a^T} \beta^{I^T}]^T$$

and λ the vector of the Lagrange multipliers,

$$\lambda = [F_I^{\gamma T} L_I^{\gamma T}]^T$$

F^{γ} and L^{γ} being the perturbations of F^{γ} and L^{γ} around their equilibrium value, the linearized dynamical equations of the tree are given under the form

$$M_{II}\ddot{x}_I + G_{II}\dot{x}_I + K_{II}x_I + K_{I2}\lambda = 0 \tag{15}$$

where M_{II} and K_{II} are symmetrical matrices, and G_{II} is a skew symmetric matrix, and the linearized constraints can then be expressed as

$$K_{2I}x_I = 0 \tag{16}$$

From the properties of Lagrange multipliers, the matrix K_{I2} is equal to K_{2I}^T and the system (5) and (6) can then be written under the more compact form

$$M\ddot{x} + G\dot{x} + Kx = 0 \tag{17}$$

where $x = [x_I^T \lambda^T]^T$, $M = M^T$, $G = -G^T$, and $K = K^T$. These properties are expected as the equations are equivalent to Lagrange equations.

Stability Analysis

The asymptotic stability of a system cannot be determined from the stability of the linearized system (17) as the matrix M is not positive definite. Nevertheless, the system is equivalent to a reduced system which can be used for stability investigations.

To obtain such a system, the linearized constraints are solved in terms of some variables; these variables are then replaced by their expression in the equations, and the Lagrange multipliers are eliminated from the remaining equations.

If the system (17) is partitioned as

$$\begin{vmatrix} M_{uu} & M_{uv} & 0 \\ M_{vu} & M_{vv} & 0 \\ 0 & 0 & 0 \end{vmatrix} \ddot{x} + \begin{vmatrix} G_{uu} & G_{uv} & 0 \\ G_{vu} & G_{vv} & 0 \\ 0 & 0 & 0 \end{vmatrix} \dot{x}$$

$$+ \begin{vmatrix} K_{uu} & K_{uv} & A_u^T \\ K_{vu} & K_{vv} & A_v^T \\ A_u & A_v & 0 \end{vmatrix} x = 0$$

with $x = [u^T v^T \lambda^T]^T$.

The matrix A_v being a regular (square) matrix (such a matrix can always be obtained as the constraints are independent), the solution of the third row provides v as a function of u,

$$v = -A_v^{-1} A_u u = Bu$$

and the equivalent system is then given by

$$M'\ddot{u} + G'\dot{u} + K'u = 0$$

with

$$M' = M_{uu} + M_{uv}B + B^T M_{vu} + B^T M_{vv} B$$

and similar expressions for G' and K'. The Hamiltonian $H = \frac{1}{2}u^T M' u + \frac{1}{2}u^T K' u$ can then be used as testing function.

If damping is introduced and if the damping is complete, the system will be asymptotically stable when H is positive definite, and the system will be unstable when H can have negative values. It is clear that the damping is not complete for freely spinning systems. In this case, integrals of motions corresponding to the constant total angular momentum have to be combined with the Hamiltonian to obtain necessary and sufficient stability conditions.[4] The completeness of damping in presence of the integrals of motion can be determined by an appropriate extension of Muller's theorem.[5,6] A complete answer to the stability problem can then be obtained and easily implemented in a computer program.

Conclusions

In this paper we developed the equations of motion for a system of interconnected deformable bodies with closed-loop configuration. A classical tree configuration was associated with the system by cutting a certain number of arcs in the loops. The equations of motion were then derived in terms of the associated tree variables and of the forces and torques in the cut arcs. In our formalism, these forces and torques are equivalent to the Lagrange multipliers corresponding to the kinematical loop constraints. As these constraints can in principle be solved, it was shown how the Lagrange equations of the system can be obtained. But for the purpose of the paper, the equations were kept under the form of Lagrange equations with constraints.

These equations permit the equilibrium synthesis by solving a system of algebraic equations. The stability of a particular equilibrium was also investigated from considerations on the linearized equations about this equilibrium. Necessary and sufficient conditions are obtained for completely damped systems. These conditions can be expressed in terms of the system parameters and are easily implemented in a computer program. An appropriate package has been added to our basic program relevant to interconnected deformable bodies.

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